Conformal and Scale Invariant Transformations Simplified

Robert D. Klauber
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Conformal Transformations

Simple definition: A conformal transformation preserves angles between lines.

Example of a transformation (which may or may not be conformal):

Consider a 2D Cartesian coordinate system painted onto on a flat elastic surface. The transformation stretches the elastic and in so doing, stretches the coordinate grid line spacing. (The gird lines can be altered, i.e., elongated and/or twisted, in any number of ways depending on how the stretching is carried out.)

Any line between two coordinate points gets altered, too. If two such lines cross at an angle, a conformal transformation (a stretching of the elastic) keeps this angle the same after the transformation (stretching) as it was before.

Example of a non-conformal transformation: Stretched horizontally only.

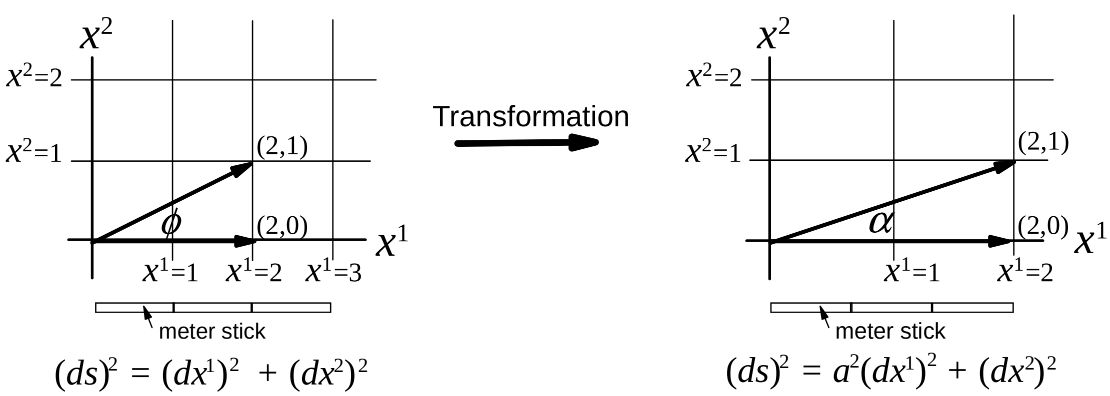

Figure1. Elastic Sheet and Its Coordinate Grid Stretched in x^1 Direction by Factor of *a*

Note ds is the physical length measured in meter sticks between two points[1]. dx^1 (and dx^2) is the coordinate difference determined by the coordinate grid numbers between those same two points.

The physical distance the point with coordinates (2,0) is from the origin on the RHS of Fig. 1 is 3 meters. So ds (Δs really) between these two coordinate points is 3 meters. It has been stretched from 2 meters on the LHS. But the coordinate difference dx^1 (Δx^1 really) between the two points on the RHS is still 2, as it was on the LHS.

Note from

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 \xrightarrow{\text{Transformation}} (ds)^2 = a^2(dx^1)^2 + (dx^2)^2 , \qquad (1)$$

where a is the scaling factor, we have, for the horizontal line with the arrow head on RHS in the example of Fig. 1,

[1] ds is sometimes called the line element.

$$\text{For } dx^2 = 0, \quad ds = adx^1 \qquad \text{where on RHS of Fig. 1} \qquad a = \frac{3}{2}. \tag{2}$$

But how about the angle between the two lines shown with arrow heads in Fig. 1? What happens to that angle as the stretching (transformation) occurs? How does ϕ on the LHS compare with α on the RHS?

Note that

$$tan\,\phi = \frac{\text{phys distance opposite side}}{\text{phys distance adjacent side}} = \frac{1}{2} = \frac{dx^2}{dx^1} \qquad tan\,\alpha = \frac{\text{phys distance opposite side}}{\text{phys distance adjacent side}} = \frac{1}{3} = \frac{dx^2}{adx^1} \quad . \tag{3}$$

$$\alpha \neq \phi$$

Thus, we see that the angle between the two lines is changed by the transformation, and this transformation is *not* conformal.

Example of a conformal transformation: Stretched horizontally and vertically by the same amount.

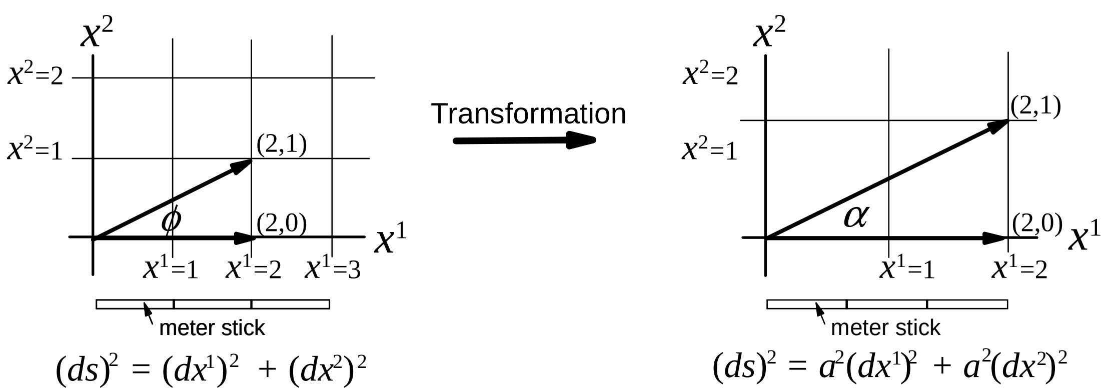

Figure 2. Elastic Sheet and Its Coordinate Grid Stretched in Both x^1 and x^2 Directions by Same Factor of *a*

Fig. 2 is similar to Fig. 1 except it is also stretched in the vertical direction by the same scaling factor *a* as in the horizontal direction. That is, the stretching is isotropic, the same in all directions. Thus, on the RHS of Fig. 2,

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 \qquad \xrightarrow{\text{Transformation}} \qquad (ds)^2 = a^2\left((dx^1)^2 + (dx^2)^2\right), \tag{4}$$

and

$$tan\,\phi = \frac{\text{phys distance opposite side}}{\text{phys distance adjacent side}} = \frac{1}{2} = \frac{dx^2}{dx^1} = .5 \qquad tan\,\alpha = \frac{\text{phys distance opposite side}}{\text{phys distance adjacent side}} = \frac{1.5}{3} = \frac{adx^2}{adx^1} = .5 \tag{5}$$

$$\alpha = \phi .$$

Hence, the isotropic transformation of Fig. 2 is conformal.

Nomenclature and Metrics

Note we can re-write (1) and (4) in terms of the metric g_{ij} (with the special case Cartesian $g_{ij} = \delta_{ij}$ = Diag (1, 1)).

$$
\begin{aligned}
&(ds)^2 = (dx^1)^2 + (dx^2)^2 \xrightarrow{\text{metric notation}} (ds)^2 = g_{ij}dx^i dx^j = \delta_{ij}dx^i dx^j = \begin{bmatrix} dx^1 & dx^2 \end{bmatrix}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix} \\
&(ds)^2 = a^2(dx^1)^2 + (dx^2)^2 \xrightarrow{\text{metric notation}} (ds)^2 = g_{ij}dx^i dx = \begin{bmatrix} dx^1 & dx^2 \end{bmatrix}\begin{bmatrix} a^2 & \\ & 1 \end{bmatrix}\begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix} \\
&(ds)^2 = a^2\left((dx^1)^2 + (dx^2)^2\right) \xrightarrow{\text{metric notation}} (ds)^2 = g_{ij}dx^i dx^j = a^2\delta_{ij}dx^i dx^j = \begin{bmatrix} dx^1 & dx^2 \end{bmatrix} a^2 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix}
\end{aligned} \tag{6}
$$

Example of Non-uniform Stretching

In the prior examples the stretching was uniform. That is, it did not vary from place to place on the elastic sheet. a was constant.

Now, consider instead a stretching that varied according to the coordinate location. That is, imagine the stretching state similar to the RHS of Fig. 1 (middle row of (6)), except that the amount of stretch varies with either x^1, x^2, or both, i.e., a is not constant, but, in general, $a = a(x^1, x^2)$.

For illustrative purposes we will take the stretching in the x^1 direction (and not in the x^2 direction), but where a is a function only of x^2. That is,

$$
(ds)^2 = a^2(dx^1)^2 + (dx^2)^2 \quad \text{with } a = a(x^2) \quad \rightarrow (ds)^2 = \begin{bmatrix} dx^1 & dx^2 \end{bmatrix} \underbrace{\begin{bmatrix} a^2(x^2) & 0 \\ 0 & 1 \end{bmatrix}}_{g_{ij}} \begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix}. \tag{7}
$$

Fig. 3 shows this graphically.

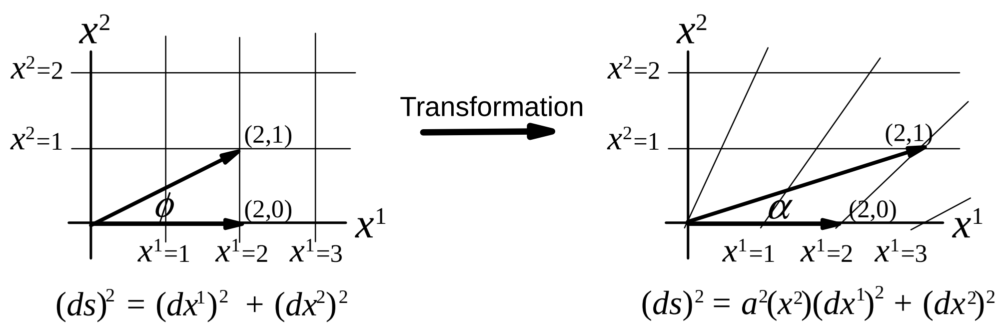

Figure 3. Example of Non-Uniform Stretching (Dependent on x^2 Coordinate)

It should be obvious from Fig. 3 that $\alpha \neq \phi$, so the transformation is not conformal.

4D: 3D of Space and 1D of Time

Consider now the Lorentzian (flat) spacetime with Minkowski coordinate line element relation (where $g_{\mu\nu} = \eta_{\mu\nu} = \text{Diag}\ (1, -1, -1, -1)$),

$$
(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \xrightarrow{\text{metric notation}} (ds)^2 = \eta_{\mu\nu}dx^\mu dx^\nu . \tag{8}
$$

A more general relation, for any 4D spacetime, which might be curved, is

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{9}$$

Cosmological Example

One solution to Einstein's field equations by Friedmann takes the form (with the units convention chosen such that $c = 1$, where $dx^0 = cdt = dt$),

$$(ds)^2 = (dt)^2 - a^2(t)\left((dx^1)^2 - (dx^2)^2 - (dx^3)^2\right) \xrightarrow{\text{metric notation}} (ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{10}$$

To get a feeling for (10), one can think in terms of an analogy with Fig. 2. The spatial 3D universe is flat in (10) like the 2D elastic sheet in Fig. 2. As time evolves, the expansion (stretching in Fig. 2) increases, and the 3D expansion is isotropic. 3D-wise it is a conformal transformation of space continually throughout time.

The spatial coordinates in the universe can be thought of as "fixed to the fabric of the universe" while the distance between coordinate points increases over time. By "fixed to the fabric of the universe" we mean given 3D coordinates (x^1, x^2, x^3) at the center of a galaxy are fixed to that center. The center of each galaxy is like a point on the coordinate grid (think point (2,1) in Fig. 2), and it keeps the same coordinate grid numbers throughout the expansion. But the physical distance (the integral of ds) between two points (two galaxies) is the number of meter sticks we would lay out between the two galaxies to measure how far apart they are. Thus, ds between two points grows over time, but dx^i stays the same.

Thinking in terms of 4D, the expansion/stretching is not isotropic, however, since no expansion/stretching is going on with the time coordinate t. It has no $a(t)$ scaling factor in front of it in (10). Thus, if we were to plot t vs x^1 (t on the vertical axis), we would get a figure like Fig. 3, with t instead of x^2. Thus, in 4D, it appears that the transformation of (10) is not isotropic and not conformal.

Note that t is defined as the time on a standard clock at any coordinate point in the universe. We assume no gravitational effects on t as most of the universe is empty space with negligible gravitational fields. t is like the usual time coordinate in a Lorentzian spacetime that we ourselves are fixed to.

A Trick Used in Cosmological Analyses

It turns out that analysis of the universe's expansion is simplified if we can modify our coordinate grid somewhat by using a different coordinate value for time than the time on standard clocks at 3D coordinate grid points.

Note that, just as Δx^1 between grid lines in any of Figs. 1 to 3 does not, on the RHS, equal one meter, we can, whenever we like, simply take different coordinate grids instead of the ones we have. For example, on the LHS we can erase the original grid lines and paint in new ones at distances other than one meter apart. We will then no longer have the identity matrix as our metric, but something else; and our coordinates differences dx^i will be different, too. But ds between the same two physical points will be the same. At the same point in time, the underlying space will not change, and distances between points on the elastic sheet (points in the centers of galaxies) will remain the same.

In cosmology, it can help analysis if we modify our time coordinate grid (and leave the spatial gird as it is) so that it no longer measures in seconds, but something else. That is, we define a new time coordinate grid, whose numeric values we designate by η (which is *not* to be confused with $\eta_{\mu\nu}$), as

$$d\eta = \frac{dt}{a(t)} \quad \rightarrow \quad dt = a(t)\,d\eta \,. \tag{11}$$

Note this changes (10) into

$$(ds)^2 = a^2(t)\left((d\eta)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\right). \tag{12}$$

Redefining our coordinate time variable gives us a 4D conformal line element relation. Points with 4D coordinates (η, x^1, x^2, x^3) will evolve conformally. Those with coordinates (t, x^1, x^2, x^3) will not. One can imagine (we won't delve into it here) that this provides great advantages in analysis. After the analysis is carried out, one simply converts the η parameter back to the t parameter to get actual time as we would measure it on standard clocks.

It should come as little surprise that η is designated as conformal time.

Note that a can be expressed as a function of t or η. For analysis, it is best to choose the latter, and (12) becomes

$$(ds)^2 = a^2(\eta)\left((d\eta)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\right). \qquad (13)$$

To find t at the end of analysis using η, we simply use (see (11))

$$t = \int dt = \int a(\eta)\,d\eta . \qquad (14)$$

Note regarding definition of conformal transformation

Our definition on page 1 of "conformal transformation" was a bit abbreviated in order to keep things simple. Implied in it, and treated explicitly in our simplified examples, we considered changes of macroscopic lines under a transformation. To be precise, a conformal transformation preserves intersection angels between local (infinitesimal) line segments at every point.

Scale Invariant Transformations

Note the similarity between the first and third rows of (6), i.e., between the unstretched LHS and isotropically stretched RHS of Fig. 2. The latter case is simply the former multiplied by a^2. In fact, the quantity

$$\left(\frac{ds}{a}\right)^2 = (dx^1)^2 + (dx^2)^2 \quad \text{is the same for any } a . \qquad (15)$$

Similarly, for 4D, as in (13)

$$\left(\frac{ds}{a}\right)^2 = (d\eta)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad \text{is the same for any } a . \qquad (16)$$

In both (15) and (16), the entire space is scaled by the same scaling factor a in all directions. This is a scaling transformation.

It turns out, in classical (non-quantum) field theory, that a scaling transformation, which results in a relation like (16), leaves the whole theory unchanged. That is, the theory is invariant under a transformation of scale. It has no characteristic length scale. We will look at an example of this shortly.

Simple definition: If the governing equations of a given theory retain the same form under a scaling transformation, that theory is a scale invariant theory.

Scale invariance and conformal invariance typically (with few exceptions) imply one another. Having one far more often than not, means having the other. (Try to imagine an angle preserving transformation that is not a stretching in all directions by the same factor a.)

Note that quantum field theories (QFTs) do not generally have scale invariance. For example, the coupling strength of the electromagnetic field α increases as one gets closer to a charge source such as an election. (This is in addition to the inverse square law with distance effect. In QED, what was a classical constant in front of that law now changes with distance between charges.) QED depends on length scale.

Further, the field equations of QFT are generally not scale invariant, unlike classical theory. An example is shown below, after the following classical field theory example.

Example of classical field theory scale invariance

Maxwell's equations in the absence of charges or currents take the form of wave equations (17), where, to make things a little clearer, we do not assume units where the speed of light c is one.

$$\frac{\partial}{\partial x^i}\frac{\partial \mathbf{E}}{\partial x^i}=\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad\qquad \frac{\partial}{\partial x^i}\frac{\partial \mathbf{B}}{\partial x^i}=\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}\,. \tag{17}$$

(17) is usually interpreted with t being standard clock time in a Cartesian coordinate system x^i. So, the grid lines x^i in this case represent physical distances measured in meter sticks and t represents time in seconds.

Now consider the scaling transformation to the 4D primed coordinate system of (18) where λ is a constant and t' and $x'^{\,i}$ are simply coordinate grid numbers and not direct indicators of seconds or meters.

$$t=\lambda t' \qquad\qquad x^i=\lambda x'^i\,. \tag{18}$$

Putting (18) into (17), we have

$$\frac{1}{\lambda^2}\frac{\partial}{\partial x'^i}\frac{\partial \mathbf{E}}{\partial x'^i}=\frac{1}{c^2}\frac{1}{\lambda^2}\frac{\partial^2\mathbf{E}}{\partial t'^2} \qquad\qquad \frac{1}{\lambda^2}\frac{\partial}{\partial x'^i}\frac{\partial \mathbf{B}}{\partial x'^i}=\frac{1}{c^2}\frac{1}{\lambda^2}\frac{\partial^2\mathbf{B}}{\partial t'^2}\,. \tag{19}$$

The λ's cancel leaving the exact same form of the wave equations in t' and $x'^{\,i}$ as in t and x^i. That is, we have the exact same theory in terms of any 4D transformed coordinates where the transformation was a scaling transformation. This is a <u>scale invariant theory.</u>

Example of a quantum field theory that is non-scale invariant

Consider the free scalar field equation of quantum field theory, i.e., the Klein-Gordon equation (where "scalar" here is unrelated to "scaling" as used above, and again, we do not assume units where c and $\hbar$ are one),

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial}{\partial x^i}\frac{\partial\phi}{\partial x^i}+\frac{m^2c^2}{\hbar}\phi=0\,. \tag{20}$$

Substitution of (18) into (20) yields

$$\frac{1}{c^2}\frac{1}{\lambda^2}\frac{\partial^2\phi}{\partial t'^2}-\frac{1}{\lambda^2}\frac{\partial}{\partial x'^{\,i}}\frac{\partial\phi}{\partial x'^{\,i}}+\frac{m^2c^2}{\hbar}\phi=0\,. \tag{21}$$

This time the λ's do not drop out and the form of the equation in scaled (primed) coordinates is not the same as for the unscaled equation (20). Thus, this is not a scale invariant theory.

Example of a free quantum field theory that is scale invariant

Suppose we had a massless scalar field, so $m = 0$ in (20). Then the λ's in (21) would cancel leaving the same equation form as in (20). Thus, a massless free scalar quantum field theory is scale invariant.

Example of interacting quantum field theory QED that is non-scale invariant

In quantum electrodynamics (QED), the governing field equations for the interaction between charged fermion fields ψ (like electron fields) and the electromagnetic fields of photons A_μ are (where we apologize for skipping back to units where $c = \hbar = 1$, as those are the more familiar units one usually sees when working in QED[2])

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi = -e\gamma^\mu A_\mu \psi \qquad \text{and} \qquad \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x^\alpha} A^\mu = -e\bar{\psi}\gamma^\mu \psi . \tag{22}$$

Using scaling transformation (18), we find (22) becomes

$$\left(\frac{1}{\lambda} i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi = -e\gamma^\mu A_\mu \psi \qquad \text{and} \qquad \frac{1}{\lambda^2}\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'^\alpha} A^\mu = -e\bar{\psi}\gamma^\mu \psi . \tag{23}$$

The λ's in (23) do not cancel, so QED is not scale invariant. This corroborates, from a different angle, what we said earlier, about QED not being a scale invariant theory.

Another aspect of scale invariant theories

A field theory can be scale invariant if we scale not just the coordinates, but the field also. That is, in addition to (18), we also scale the field by the parameter λ raised to some power d, i.e., (where ϕ represents any field and the prime does not mean differentiation)

$$\phi' = \lambda^d \phi . \tag{24}$$

In (18) and (19), $d = 0$. The massless case we examined in (21) also had $d = 0$. In the more general case, d of (24) need not be zero to have a scale invariant theory.

In a scale invariant theory having a solution to the field equation as $\phi(x^\mu)$, $\lambda^d \phi(\lambda x^\mu)$ is also a solution.

Conformal quantum field theories

In quantum field theory, the terms scale-invariant field theory and conformally invariant field theory are typically used interchangeably, and generally referred to as simply a conformal field theory (CFT).

Note that CFTs often have a value for d that differs from the corresponding d of the classical field theory. The additional contributions to d appearing in CFTs are known as anomalous scaling dimensions[3].

Final note on general sense of scale invariance

We have focused on scale invariance for dimensions of space and time. A theory can also be termed scale invariant if it is unchanged with respect to other variables such as energy. For energy, the theory would look the same at any energy level. This is generally not true of QFTs, as the coupling parameters for the e/m, weak, and strong forces depend on energy level. The theories of these interactions are not energy scale invariant (They are not conformal field theories.) Given the inverse relation between momentum/distance and energy/time, this should not be surprising, since these theories are not scale invariant with respect to spacetime.

Further, an object (rather than laws, i.e., theory) can also be considered scale invariant if some feature of it does not change as length, or other variable, is scaled.

[2] See R.D. Klauber, *Student Friendly Quantum Field Theory*, 2nd ed. (Sandtrove 2013), eqs. (7-18) and (7-19), pg. 186.
[3] The term "anomalous" typically refers to disparities between corresponding classical and quantum field theories.